



PAPER-1(B.E./B. TECH.)

JEE (Main) 2021

Questions & Solutions

(Reproduced from memory retention)

Date : 26 February, 2021 (SHIFT-2) Time ; (3.00 pm to 06.00 pm)

Duration : 3 Hours | Max. Marks : 300

SUBJECT : MATHEMATICS

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MATHEMATICS

1. If $0 < a, b < 1$, $\tan^{-1}a + \tan^{-1}b = \frac{\pi}{4}$ find value of $(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) \dots \infty$

- (1) $\ln 3$ (2) $2 \ln 2$ (3) $\ln 2$ (4) $\ln 2 + \ln 3$

Ans. (3)

Sol. $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \frac{\pi}{4} \Rightarrow a+b = 1-ab \Rightarrow (1+a)(1+b) = 2$

Now, $a+b - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) \dots \infty$
 $= \left(a - \frac{a^2}{2} + \frac{a^3}{3} \dots\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} \dots\right)$
 $= \ln(1+a) + \ln(1+b) = \ln(1+a)(1+b) = \ln 2$

2. If $f(x) = \int_0^x e^t f(t) dt + e^x$, then $f(x)$ is equal to

- (1) $2e^{e^x-1}$ (2) $2e^{e^x-1} + 1$ (3) $e^{e^x-1} + 1$ (4) $2e^{e^x-1} - 1$

Ans. (4)

Sol. $f'(x) = e^x \cdot f(x) + e^x$
 $\Rightarrow \frac{f'(x)}{f(x)+1} = e^x \Rightarrow \ln(f(x)+1) = e^x + c$

put $x = 0$

$\ln 2 = 1 + c$

$\therefore \ln(f(x)+1) = e^x + \ln 2 - 1$

$\Rightarrow f(x)+1 = 2 \cdot e^{e^x-1} \Rightarrow f(x) = 2e^{e^x-1} - 1$

3. A seven digit number has been formed by using digit 3, 3, 4, 4, 4, 1, 1 (by taking all at a time). Probability that number is even.

- (1) $\frac{2}{7}$ (2) $\frac{3}{7}$ (3) $\frac{5}{14}$ (4) $\frac{3}{14}$

Ans. (2)

Sol. $n(S) = \frac{7!}{2!3!2!}$
 $n(E) = \frac{6!}{2!2!2!}$
 $P(E) = \frac{n(E)}{n(S)} = \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!}$
 $= \frac{1}{7} \times 3 = \frac{3}{7}$

4. If A_1 & A_2 are area bounded by :
 A_1 : $y = \sin x$, $y = \cos x$ & y -axis in Ist quadrant
 A_2 : $y = \sin x$, $y = \cos x$ & x -axis & $x = \frac{\pi}{2}$.

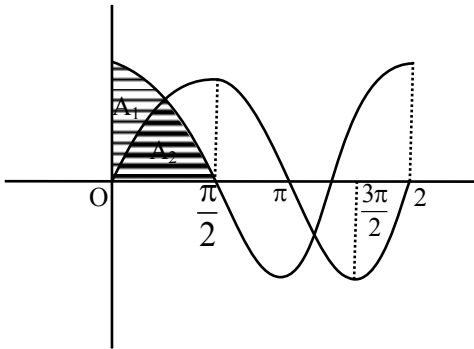
Then

(1) $A_1 : A_2 = 1 : \sqrt{2}$; $A_1 + A_2 = 1$
 (3) $A_1 : A_2 = 1 : 2$; $A_1 + A_2 = 2$

(2) $A_1 : A_2 = \sqrt{2} : 1$; $A_1 + A_2 = \sqrt{2} + 1$
 (4) $A_1 : A_2 = 1 : 2$; $A_1 + A_2 = 1$

Ans. (1)

Sol. $A_1 + A_2 = \int_0^{\pi/2} \cos x \cdot dx = \sin x \Big|_0^{\pi/2} = 1$



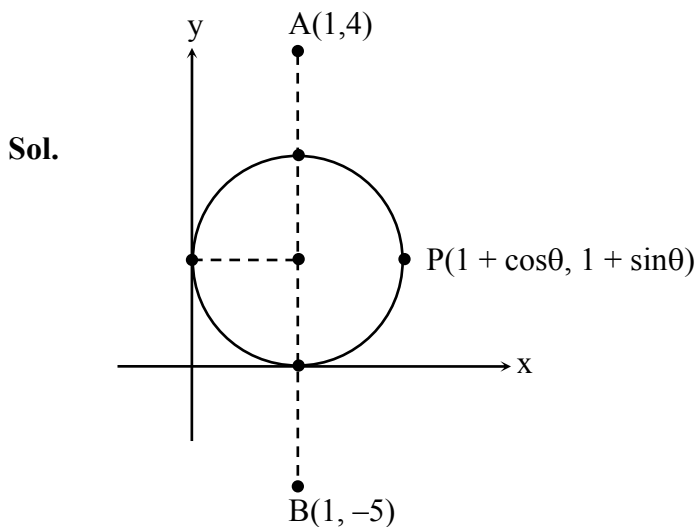
$$A_1 = \int_0^{\pi/4} (\cos x - \sin x) dx = (\sin x + \cos x) \Big|_0^{\pi/4} = \sqrt{2} - 1$$

$$\therefore A_2 = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$$

$$\therefore \frac{A_1}{A_2} = \frac{\sqrt{2}-1}{2-\sqrt{2}} = 1 : \sqrt{2}$$

5. Consider the circle $(x - 1)^2 + (y - 1)^2 = 1$; A(1, 4) B(1, -5). If P is a point on circle. Such that (PA) + (PB) is maximum, then P, A, B lie on ?
 (1) ellipse (2) hyperbola (3) Straight line (4) None of these

Ans. (3)



$$\therefore PA^2 = \cos^2 \theta + (\sin \theta - 3)^2 = 10 - 6 \sin \theta$$

$$PB^2 = \cos^2 \theta + (\sin \theta - 6)^2 = 37 - 12 \sin \theta$$

$$PA^2 + PB^2 = 47 - 18 \sin \theta|_{\max} \Rightarrow \theta = \frac{3\pi}{2}$$

$\therefore$ P, A, B lie on a line $x = 1$

6. If A triangle is inscribed in a circle of radius r , then which of the following triangle can have maximum area

(1) equilateral triangle with height $\frac{2r}{3}$

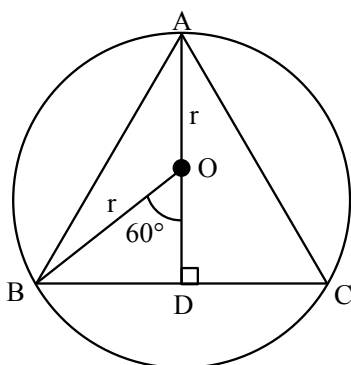
(2) right angle triangle with side $2r, r$

(3) equilateral triangle with side $\sqrt{3}r$

(4) isosceles triangle with base $2r$

Ans. (3)

Sol.



$$OD = r \cos 60^\circ = \frac{r}{2}$$

$$\text{Height} = AD = \frac{3r}{2}$$

$$\text{Now } \sin 60^\circ = \frac{\frac{3}{2}r}{AB}$$

$$\Rightarrow AB = \sqrt{3}r$$

7. If $f(a) = 2$ and $f(a) = 4$, then find value of $\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}$

(1) $4 - 2a$

(2) $2a - 4$

(3) 0

(4) $a - 4$

Ans. (1)

Sol. By L-H rule

$$L = \lim_{x \rightarrow a} \frac{f(a) - af'(x)}{1}$$

$$\therefore L = 4 - a \quad (2)$$

8. If $\vec{a}_1 = x\hat{i} - \hat{j} + \hat{k}$, $\vec{a}_2 = \hat{i} + y\hat{j} + z\hat{k}$ are collinear vector, then a unit vector which is parallel to $x\hat{i} + y\hat{j} + z\hat{k}$, can be

(1) $\frac{1}{\sqrt{3}} (\hat{i} - \hat{j} + \hat{k})$ (2) $\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ (3) $\frac{1}{\sqrt{2}} (\hat{j} - \hat{k})$ (4) $\frac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$

Ans. (1)

Sol. $\frac{x}{1} = -\frac{1}{y} = \frac{1}{z} = \lambda$ (let)

Unit vector parallel to $x\hat{i} + y\hat{j} + z\hat{k} = \pm \frac{\left(\lambda\hat{i} - \frac{1}{\lambda}\hat{j} + \frac{1}{\lambda}\hat{k}\right)}{\sqrt{\lambda^2 + \frac{2}{\lambda^2}}}$

for $\lambda = 1$ it is $\pm \frac{(\hat{i} - \hat{j} + \hat{k})}{\sqrt{3}}$

9. If $y + z = 5$, $\frac{1}{y} + \frac{1}{z} = \frac{5}{6}$, $y > z$

If prime factorization of a natural number $N = 2^x 3^y 5^z$. Find number of odd divisors of N including 1.

(1) $6x$ (2) 6 (3) 12 (4) 11

Ans. (3)

Sol. Solving given two equation we get $y = 3$, $z = 2$

$\Rightarrow N = 2^x 3^3 5^2$

number of odd divisor = $(2 + 1)(3 + 1) = 12$

10. If a curve $y = f(x)$ is given by $\frac{dy}{dx} = \frac{xy^2 + y}{x}$ passing through $(-2, 3)$ meets the line $L = 0$ at $(3, y)$ then y is

(1) $-\frac{11}{19}$ (2) $-\frac{18}{19}$ (3) $-\frac{11}{29}$ (4) $\frac{11}{19}$

Ans. (2)

Sol. $\frac{dy}{dx} = \frac{xy^2 + y}{x}$

$\Rightarrow \frac{xdy - ydx}{y^2} = x dx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + C$$

It passes through $(-2, 3)$

$$\Rightarrow \frac{2}{3} = 2 + C$$

$$\Rightarrow C = \frac{-4}{3}$$

$$\therefore \text{curve is } \frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$$

It also passes through $(3, y)$

$$\frac{-3}{y} = \frac{9}{2} - \frac{4}{3}$$

$$\Rightarrow \frac{-3}{y} = \frac{19}{6}$$

$$\Rightarrow y = -\frac{18}{19}$$

11. If $f(x) = \int_1^x \frac{\log_e(t)}{(1+t)} dt$, then $f(e) + f\left(\frac{1}{e}\right)$ is

(1) 0

(2) 1

(3) -1

(4) $\frac{1}{2}$

Ans. (4)

Sol. $f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ln t}{1+t} dt + \int_1^{1/e} \frac{\ln t}{1+t} dt = I_1 + I_2$

$$I_2 = \int_1^{1/e} \frac{\ln t}{1+t} dt \quad \text{put } t = \frac{1}{z} \quad dt = -\frac{dz}{z^2}$$

$$= \int_1^e -\frac{\ln z}{1+\frac{1}{z}} \times \left(-\frac{dz}{z^2}\right) = \int_1^e \frac{\ln z}{z(z+1)} dz$$

$$f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\ln t}{1+t} dt + \int_1^{1/e} \frac{\ln t}{t(t+1)} dt = \int_1^e \frac{\ln t}{1+t} + \frac{\ln t}{t(t+1)} dt$$

$$= \int_1^e \frac{\ln t}{t} dt = \int_0^1 u du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}$$

12. Consider the system of equation $x + 2y - 3z = a$, $2x + 6y - 11z = b$, $x - 2y + 7z = c$ then

- (1) unique solution for $\forall a, b, c$ (2) infinite solution for $5a = 2b + c$
(3) no solution for all a, b, c (4) unique solution for $5a = 2b + c$

Ans. (2)

Sol. $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$

$$= 20 - 2(25) - 3(-10)$$

$$= 20 - 50 + 30 = 0$$

$$D_1 = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$$

$$= 20a - 2(7b + 11c) - 3(-2b - 6c)$$

$$= 20a - 14b - 22c + 6b + 18c$$

$$= 20a - 8b - 4c$$

$$= 4(5a - 2b - c)$$

$$D_2 = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$$

$$= 7b + 11c - a(25) - 3(2c - b)$$

$$= 7b + 11c - 25a - 6c + 3b$$

$$= -25a + 10b + 5c$$

$$= -5(5a - 2b - c)$$

$$D_3 = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$$

$$= 6c + 2b - 2(2c - b) - 10a$$

$$= -10a + 4b + 2c$$

$$= -2(5a - 2b - c)$$

for infinite solution

$$D = D_1 = D_2 = D_3 = 0$$

$$\Rightarrow 5a = 2b + c$$

13. A function $f(k)$ is defined A to A where $A = \{1, 2, 3, 4, 5, \dots, 10\}$, such that

$$f(k) = \begin{cases} k+1, & k \in \text{odd} \\ k, & k \in \text{even} \end{cases}$$

If $\text{gof}(x) = f(x)$ then number of mapping of $g(x)$ from $A \rightarrow A$ is

- (1) $^{10}C_5$ (2) 10^5 (3) 5^5 (4) $5!$

Ans. (2)

Sol. $g(f(x)) = f(x)$

$\Rightarrow g(x) = x$, when x is even.

$\therefore$ So total number of functions from A to A
 $= 10^5 \times 1 = 10^5$

14. If $F_1(A, B, C) = (\sim A \vee B) \vee (\sim A) \vee (\sim C \wedge (A \vee B))$

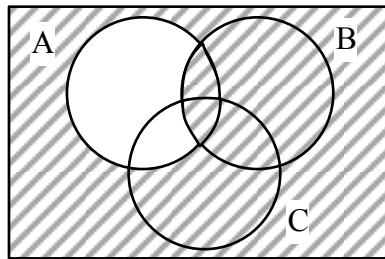
$F_2(A, B, C) = (A \vee B) \vee (A \rightarrow \sim B)$

Then which of the following is true :

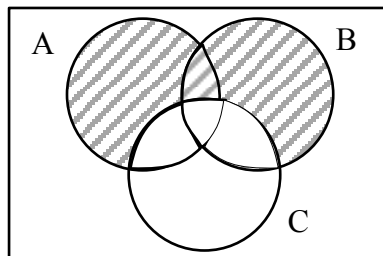
- (1) F_1 is Tautology and F_2 is Tautology
- (2) F_1 is Tautology and F_2 is not Tautology
- (3) F_1 is not Tautology and F_2 is Tautology
- (4) Neither is Tautology

Ans. (3)

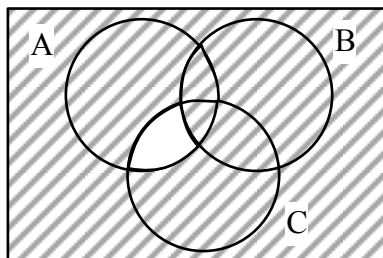
Sol. $(\sim A \vee B) \equiv$



$\sim C \wedge (A \vee B)$

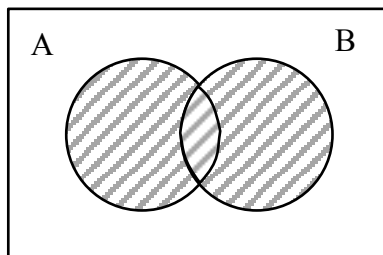


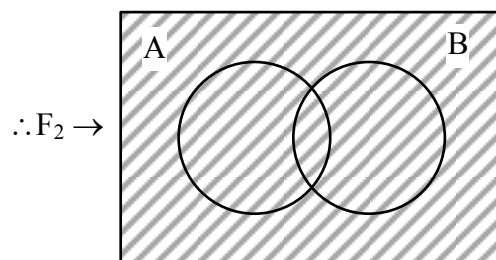
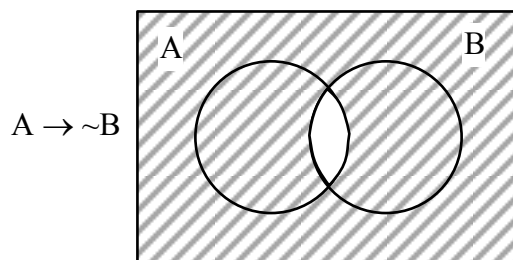
$\therefore F_1 :$



$\Rightarrow$ Not tautology

$A \vee B \equiv$





Tautology

Truth table for F_1 (A, B, C)

A	B	C	$\sim A$	$\sim C$	$A \vee B$	$\sim A \vee B$	$\sim C \wedge (A \vee B)$	$(\sim A \vee B) \vee (\sim C \wedge (A \vee B)) \vee \sim A$
T	T	T	F	F	T	T	F	T
T	F	F	F	T	T	F	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	F	F	F
F	T	T	T	F	T	T	F	T
F	F	F	T	T	F	T	F	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	F	T	F	T

Truth table for F_2

A	B	$A \vee B$	$\sim B$	$A \rightarrow \sim B$	$(A \vee B) \vee (A \rightarrow \sim B)$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	T	T
F	F	F	T	T	T

F_1 not shows tautology and F_2 shows tautology.

15. $\sum_{n=1}^{\infty} \frac{n^2 + 6n + 10}{(2n+1)!} =$

(1) $\frac{41e}{8} + \frac{19e}{8} - 10$

(2) $\frac{41e}{8} + \frac{19}{e} - 10$

(3) $\frac{41}{8}e - \frac{19}{8e} - 10$

(4) $\frac{41}{8}e - \frac{19}{8e} - 80$

Ans. (3)

Sol. $\sum_{n=1}^{\infty} \frac{n^2 + 6n + 10}{(2n+1)!}$

put $2n + 1 = r$, where $r = 3, 5, 7, \dots$

$$\Rightarrow n = \frac{r-1}{2}$$

$$\frac{n^2 - 6n + 10}{(2n+1)!} = \frac{\left(\frac{r-1}{2}\right)^2 + 3r - 3 + 10}{r!} = \frac{r^2 + 10r + 29}{4r!}$$

$$\begin{aligned} \text{Now } \sum_{r=3,5,7} \frac{r(r-1) + 11r + 29}{4r!} &= \frac{1}{4} \sum_{r=3,5,7} \left(\frac{1}{(r-2)!} + \frac{11}{(r-1)!} + \frac{29}{r!} \right) \\ &= \frac{1}{4} \left\{ \left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots \right) + 11 \left(\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots \right) + 29 \left(\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots \right) \right\} \\ &= \frac{1}{4} \left\{ \frac{e - \frac{1}{e}}{2} + 11 \left(\frac{e + \frac{1}{e} - 2}{2} \right) + 29 \left(\frac{e - \frac{1}{e} - 2}{2} \right) \right\} \\ &= \frac{1}{8} \left\{ e - \frac{1}{e} + 11e + \frac{11}{e} - 22 + 29e - \frac{29}{e} - 58 \right\} \\ &= \frac{1}{8} \left\{ 41e - \frac{19}{e} - 80 \right\} \end{aligned}$$

- 16.** Foot of the perpendicular from the points $(3, 4, 1)$ on the line of intersection of the planes $x + 2y + z - 6 = 0$ & $y + 2z = 4$ is

(1) $\left(\frac{10}{7}, \frac{12}{7}, \frac{8}{7}\right)$ (2) $\left(\frac{10}{7}, \frac{-12}{7}, \frac{8}{7}\right)$ (3) $\left(\frac{10}{7}, \frac{-12}{7}, \frac{-8}{7}\right)$ (4) $\left(\frac{-10}{7}, \frac{12}{7}, \frac{8}{7}\right)$

Ans. (1)

Sol. Let D.R's of line are a, b, c

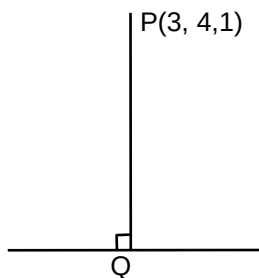
$$\therefore a + 2b + c = 0$$

$$0.a + b + 2c = 0$$

$$\frac{a}{3} = \frac{b}{-2} = \frac{c}{1}$$

Points on the line is $(-2, 4, 0)$

$$\therefore \text{equation of line is } \frac{x+2}{3} = \frac{y-4}{-2} = \frac{z}{1} = t$$



Point Q on the line is $(3\lambda - 2, -2\lambda + 4, \lambda)$

DR's of PQ ; $3\lambda - 5, -2\lambda, \lambda - 1$

DR's of y lines are 3, -2, 1

Since $PQ \perp$ line $\Rightarrow 3(3\lambda - 5) - 2(-2\lambda) + 1(\lambda - 1) = 0$

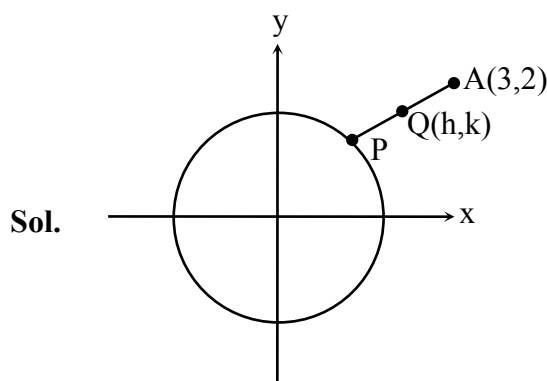
$$\Rightarrow 14\lambda - 16 \Rightarrow \lambda = \frac{8}{7}$$

$$\therefore Q\left(\frac{10}{7}, \frac{12}{7}, \frac{8}{7}\right)$$

17. From the point A(3, 2), a line is drawn to any point on the circle $x^2 + y^2 = 1$. if locus of midpoint of this line segment is a circle, the its radius is

- (1) $\frac{\sqrt{13}}{2}$ (2) $\frac{1}{2}$ (3) $\frac{\sqrt{11}}{2}$ (4) $\sqrt{11}$

Ans. (1)



$\therefore P \equiv (2h - 3, 2k - 2) \rightarrow$ on circle

$$\therefore \left(h - \frac{3}{2}\right)^2 + (k - 1)^2 = \frac{1}{4}$$

$$\Rightarrow \text{radius} = \frac{1}{2}$$

18. Given $f(x) = \sin^{-1}x$, $g(x) = \frac{x^2 - x - 2}{2x^2 - x - 6}$, $x \neq 2$ and $g(2) = \lim_{x \rightarrow 2} g(x)$ find domain $\text{fog}(x)$

- (1) $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$ (2) $(-\infty, -1] \cup [2, \infty)$
(3) $\left[-2, -\frac{4}{3}\right]$ (4) $(-\infty, 2)$

Ans. (1)

Sol. $g(2) = \lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$

For domain of fog (x)

$$\left| \frac{x^2 - x - 2}{2x^2 - x - 6} \right| \leq 1 \Rightarrow (x+1)^2 \leq (2x+3)^2$$

$$\Rightarrow (3x+4)(x+2) \geq 0$$

$$x \in (-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right]$$

19. If $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right) & ; \quad x < -1 \\ |ax^2 + x + b| & ; \quad -1 \leq x < 1 \text{ is continuous } \forall x \in \mathbb{R}, \text{ then find } (a+b) \\ \sin(\pi x) & ; \quad 1 \leq x \end{cases}$

(1) -1

(2) 1

(3) 2

(4) -2

Ans. (1)

Sol. If f is continuous at $x = -1$, then

$$f(-1^-) = f(-1)$$

$$\Rightarrow 2 = |a - 1 + b|$$

$$\Rightarrow |a + b - 1| = 2 \quad \dots (i)$$

similarly

$$f(1^-) = f(1)$$

$$\Rightarrow |a + b + 1| = 0$$

$$\Rightarrow a + b = -1$$

20. If $I_{m,n} = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx$, $m, n \geq 1$ and $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} = \alpha I_{m,n}$, then find ' α '

(1) 1

(2) 2

(3) 0

(4) -1

Ans. (1)

Sol. $I_{m,n} = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx$ put $x = \frac{1}{y+1}$

$$I_{m,n} = \int_0^1 \frac{y^{n-1}}{(y+1)^{m+n}} (-1) dy = \int_0^\infty \frac{y^{n-1}}{(y+1)^{m+n}} dy \quad \dots (i)$$

Similarly $I_{m,n} = \int_0^1 x^{n-1} \cdot (1-x)^{m-1} dx$

$$\Rightarrow I_{m,n} = \int_0^{\infty} \frac{y^{m-1}}{(y+1)^{m+n}} dy \quad \dots(ii)$$

From (i) & (ii)

$$2I_{m,n} = \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy$$

$$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy + \int_1^{\infty} \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy$$

Put $y = \frac{1}{z}$

$$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy + \int_1^{\infty} \frac{z^{m-1} + z^{n-1}}{(z+1)^{m+n}} dz$$

$$\Rightarrow I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+1}} dy \Rightarrow \alpha = 1$$

21. If slope of common tangent to curve $4x^2 + 9y^2 = 36$ and $4x^2 + 4y^2 = 31$ is m then m^2 is equal to

Ans. 3

Sol. E : $\frac{x^2}{9} + \frac{y^2}{4} = 1$

C : $x^2 + y^2 = \frac{31}{4}$

equation of tangent to ellipse

$$y = mx \pm \sqrt{9m^2 + 4} \quad \dots(i)$$

equation of tangent to circle

$$y = mx \pm \sqrt{\frac{31}{4}m^2 + \frac{31}{4}} \quad \dots(ii)$$

Comparing equation (i) & (ii)

$$9m^2 + 4 = \frac{31m^2}{4} + \frac{31}{4}$$

$$\Rightarrow 36m^2 + 16 = 31m^2 + 31$$

$$\Rightarrow 5m^2 = 15$$

$$\Rightarrow m^2 = 3$$

22. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ and $A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then find $(\alpha - \beta)$

Ans. 4

Sol. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$A^{19} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix}, A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{L.H.S} = A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1+\alpha+\beta & 0 & 0 \\ 0 & 2^{20} + \alpha 2^{19} + 2\beta & 0 \\ 3\alpha+3\beta & 0 & 1-\alpha-\beta \end{bmatrix}$$

$$\text{R.H.S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \alpha + \beta = 0 \text{ and}$$

$$2^{20} + \alpha 2^{19} + 2\beta = 4$$

$$2^{20} + \alpha(2^{19} - 2) = 4$$

$$2 = \frac{4 - 2^{20}}{2^{19} - 2} = -2$$

$$\beta = 2 \Rightarrow (\alpha - \beta) = 4$$

23. In normal drawn to the curve at any point passes through a fixed point (a, b). The curve passes through (3, -3) & (4, $-2\sqrt{2}$) such that $a - 2\sqrt{2}b = 3$. Find $a^2 + b^2 + ab$

Ans. 9

Sol. Let the equation of normal is $Y - y = -\frac{1}{m}(X - x)$

Satisfy (a, b) in it $b - y = -\frac{1}{m}(a - x)$

$$\Rightarrow (b - y) dy = (x - a) dx$$

$$by - \frac{y^2}{2} = \frac{x^2}{2} - ax + c \quad \dots\dots(i)$$

It passes through (3, -3) & (4, $-2\sqrt{2}$)

$$\therefore -3b - \frac{9}{2} = \frac{9}{2} - 3a + c$$

$$\Rightarrow -6b - 9 = 9 - 6a + 2c$$

$$\Rightarrow 6a - 6b - 2c = 18$$

$$\Rightarrow 3a - 3b - c = 9 \quad \dots\dots(ii)$$

Also

$$-2\sqrt{2}b - 4 = 8 - 4a + c$$

$$4a - 2\sqrt{2}b - c = 12 \quad \dots\dots(iii)$$

$$\text{Also } a - 2\sqrt{2}b = 3 \quad \dots\dots(iv) \quad (\text{given})$$

$$(ii) - (iii) \Rightarrow -a + (2\sqrt{2} - 3)b = -3 \quad \dots\dots(v)$$

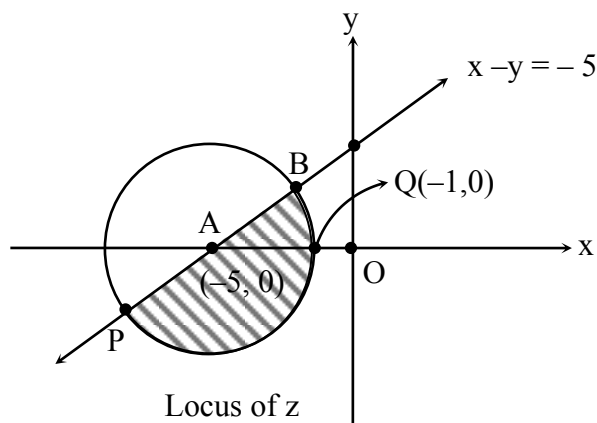
$$(iv) + (v) \Rightarrow b = 0 \quad a = 3$$

$$\therefore a^2 + b^2 + ab = 9$$

24. Let $z(z \in \mathbb{C})$ satisfy $|z + 5| \leq 5$ and $z(1 + i) + \bar{z}(1 - i) \geq -10$ if the maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$; then find $\alpha + \beta$

Ans. 48

Sol.



$$\therefore P \equiv (-5 - 2\sqrt{2}, -2\sqrt{2})$$

$$\therefore (PQ)^2 \Big|_{\max} = 32 + 16\sqrt{2}$$

$$\alpha = 32$$

$$\beta = 16$$

$$\alpha + \beta = 48$$

- 25.** $-16, 8, -4, 2, \dots$ is sequence, whose AM & GM of p^{th} & q^{th} term are the roots of $4x^2 - 9x + 5 = 0$,
Find $p + q$

Ans. 10

Sol. $-16, 8, -4, 2, \dots$

$$p^{\text{th}} \text{ term } t_p = -16 \left(\frac{-1}{2} \right)^{p-1}$$

$$q^{\text{th}} \text{ term } t_q = -16 \left(\frac{-1}{2} \right)^{q-1}$$

$$\text{Now } \frac{t_p + t_q}{2} = \frac{5}{4} \text{ \& } \sqrt{t_p t_q} = 1$$

$$\Rightarrow 16^2 \left(-\frac{1}{2} \right)^{p+q-2} = 1$$

$$\Rightarrow 2^8 = (-2)^{(p+q-2)}$$

$$\Rightarrow p + q = 10$$

- 26.** If $x_1, x_2, x_3, \dots, x_{18}$ are 18 terms and $\sum_{i=1}^{18} (x_i - \alpha) = 36$, $\sum_{i=1}^{18} (x_i - \beta)^2 = 90$ variance $(\sigma^2) = 1$ given,
then find $|\beta - \alpha|$

Ans. 4

Sol. $\sum x_i - 18\alpha = 36$

$$\sum x_i = 18(\alpha + 2) \quad \dots(i)$$

$$\sum x_i^2 + 18\beta^2 - 2\beta \sum x_i = 90$$

$$\sum x_i^2 + 18\beta^2 - 2\beta \times 18(\alpha + 2) = 90$$

$$\sum x_i^2 = 90 - 18\beta^2 + 36\beta(\alpha + 2) \quad \dots(ii)$$

$$\sigma^2 = 1 \Rightarrow \frac{1}{18} \sum x_i^2 - \left(\frac{\sum x_i}{18} \right)^2 = 1$$

$$\Rightarrow \frac{1}{18} (90 - 18\beta^2 + 36\alpha\beta + 72\beta) - \left(\frac{18(\alpha + 2)}{18} \right)^2 = 1$$

$$\Rightarrow 90 - 18\beta^2 + 36\alpha\beta + 72\beta - 18(\alpha + 2)^2 = 18$$

$$\begin{aligned} \Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - (\alpha + 2)^2 &= 1 \\ \Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4 - 4\alpha &= 1 \\ -\alpha^2 - \beta^2 + 2\alpha\beta + 4\beta - 4\alpha &= 0 \\ -(\alpha - \beta)^2 - 4(\alpha - \beta) &= 0 \\ -(\alpha - \beta)(\alpha - \beta + 4) &= 0 \\ \Rightarrow \alpha - \beta &= -4 & (\alpha \neq \beta) \\ |\beta - \alpha| &= 4 \end{aligned}$$

27. It $P_n = \alpha^n + \beta^n$, $\alpha + \beta = 1$, $\alpha\beta = -1$, $P_{n-1} = 11$, $P_{n+1} = 29$ find $(P_n)^2$ (where $n \in \mathbb{N}$)

Ans. 324

Sol. Quadratic Equation whose roots are $\alpha, \beta : x^2 - x - 1 = 0$

$$\therefore \alpha^2 = \alpha + 1 \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

$$\beta^2 = \beta + 1 \Rightarrow \beta^n = \beta^{n-1} + \beta^{n-2}$$

$$\therefore P_n = P_{n-1} + P_{n-2}$$

$$\Rightarrow P_{n+1} = P_n + P_{n-1}$$

$$\Rightarrow 29 = P_n + 11 \Rightarrow P_n = 18$$

$$\Rightarrow (P_n)^2 = 324$$

28. The number of four digit numbers whose HCF with 18 is 3 equals

Ans. 1000

Sol. Number must be an odd multiple of 3 and not a multiple of 9

4-digit odd multiples of 3 are

$$1005, 1011, \dots, 9999 \rightarrow 1499$$

4-digit odd multiples of 9 are

$$1017, 1035, \dots, 9999 \rightarrow 499$$

$$\therefore \text{Required numbers} \rightarrow 1000$$

29. Image of a point $(1, 0, -1)$ in the plane $4x - 5y + 2z = 8$ is (α, β, γ) . Find $15(\alpha + \beta + \gamma)$

Ans. 4

Sol.
$$\frac{x-1}{4} = \frac{y-0}{-5} = \frac{z+1}{2} = \frac{-2(-6)}{16+25+4} = \frac{12}{45} = \frac{4}{15}$$

$$x - 1 = \frac{16}{15} \Rightarrow x = \frac{31}{15}$$

$$y = -\frac{4}{3}$$

$$z + 1 = \frac{8}{15} \Rightarrow z = -\frac{7}{15}$$

$$\alpha = \frac{31}{15}, \beta = -\frac{4}{3}, \gamma = -\frac{7}{15}$$

$$15(\alpha + \beta + \gamma) = \left(\frac{31}{15} - \frac{4}{3} - \frac{7}{15} \right) \times 15 = 4$$

- 30.** If $f(x) = 2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ and all real roots of $f(x)$ lie in the interval $(-\alpha, -\alpha+1)$ then ' α ' is :

Ans. 2

Sol. $f(-1) = 3 > 0$

$$\begin{aligned} f(-2) &= -64 + 80 - 80 + 40 - 20 + 10 \\ &= -34 < 0 \end{aligned}$$

$\therefore$ At least one root in $(-2, -1)$

$$f(x) = 10(x^4 + 2x^3 + 3x^2 + 2x + 1)$$

$$= 10 \left(x^2 + \frac{1}{x^2} + 2 \left(x + \frac{1}{x} \right) + 3 \right)$$

$$= 10 \left(\left(x + \frac{1}{x} \right)^2 + 2 \left(x + \frac{1}{x} \right) + 1 \right)$$

$$= 10 \left(\left(x + \frac{1}{x} \right) + 1 \right)^2 > 0; \forall x \in \mathbb{R}$$

$\therefore$ Exactly one real root in $(-2, -1)$